

POSITROIDS AND SCHUBERT MATROIDS

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ABSTRACT. Recently Postnikov gave a combinatorial description of the cells in a totally-nonnegative Grassmannian. These cells correspond to a special class of matroids called positroids. We prove his conjecture that a positroid is exactly an intersection of permuted Schubert matroids. This leads to a combinatorial description of positroids that is easily computable. The main proof is purely combinatorial, using only the characteristics of a \mathcal{J} -graph.

1. INTRODUCTION

A *positroid* is a matroid that can be represented by a $k \times n$ matrix with nonnegative maximal minors. The classical theory of total positivity concerns matrices in which all minors are non-negative, and this subject was extended by Lusztig (cf. [L]).

Lusztig introduced the totally non-negative variety $G \geq 0$ in an arbitrary reductive group G and the totally non-negative part $(G/P)_{\geq 0}$ of a real flag variety (G/P) . He also conjectured that $(G/P)_{\geq 0}$ is made up of cells, and this was proved by Rietsch (cf. [R]).

In this paper, we will restrict our attention to $(Gr_{kn})_{\geq 0}$, the *totally non-negative Grassmannian*. Then there is a more refined decomposition using matroid strata. Recently, Postnikov (cf. [P]) obtained a relationship between $(Gr_{kn})_{\geq 0}$ and certain planar bicolored graphs, producing a combinatorially explicit cell decomposition of $(Gr_{kn})_{\geq 0}$. The cells correspond to positroids.

And one of the results of [P] is that each cell is an intersection of $(Gr_{kn})_{\geq 0}$ and Schubert cells corresponding to a combinatorial object called Grassmann necklace. This result implies that each positroid is included in an intersection of cyclically shifted Schubert matroids. We extend this result: each positroid is exactly an intersection of certain cyclically shifted Schubert matroids.

A more detailed formulation of the main result follows. Let $[n] := \{1, \dots, n\}$ and let $\binom{[n]}{k}$ be the collection of all k -element subsets in $[n]$. Fix some $t \in [n]$. We define the ordering $<_t$ on $[n]$ by the total order $t <_t t+1 <_t \dots <_t n <_t 1 <_t \dots <_t t-1$. For $I, J \in \binom{[n]}{k}$, where

$$I = \{i_1, \dots, i_k\}, i_1 <_t i_2 <_t \dots <_t i_k$$

and

$$J = \{j_1, \dots, j_k\}, j_1 <_t j_2 <_t \dots <_t j_k.$$

Then we set

$$I \leq_t J \text{ if and only if } i_1 \leq_t j_1, \dots, i_k \leq_t j_k.$$

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Then for $I \in \binom{[n]}{k}$ and $w \in S_n$, we define the cyclically shifted Schubert matroid as

$$SM_I^{c^{t-1}} = \{J \in \binom{[n]}{k} \mid I \leq_t J\}.$$

Then we will show that a matroid $\mathcal{M} \subseteq \binom{[n]}{k}$ is a positroid if and only if it has the form $SM_{I_1}^{c^0} \cap SM_{I_2}^{c^1} \cap \cdots \cap SM_{I_n}^{c^{n-1}}$ for a Grassmann necklace $\mathcal{I} = (I_1, \dots, I_n)$, $I_1, \dots, I_n \in \binom{[n]}{k}$. Our proof is purely combinatorial.

The paper is organized as follows. In section 2, we go over the basics of matroids and the totally nonnegative Grassmannian. In section 3, we review \mathcal{J} -diagrams and \mathcal{J} -graphs. In section 4, we give the proof of our main result. In section 5, we use the result to derive an example. In section 6, we introduce the upper Grassmann necklace. In section 7, we look at lattice path matroids in terms of positroids. In section 8, we show some related problems.

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2. PRELIMINARIES AND THE MAIN RESULT

We would like to guide the readers unfamiliar with basics in this section to [BLSWZ], [S], [F]. [P] contains more detailed description of the contents of this section.

An element in the Grassmannian Gr_{kn} can be understood as a collection of n vectors $v_1, \dots, v_n \in \mathbb{R}^k$ spanning the space \mathbb{R}^k modulo the simultaneous action of GL_k on the vectors. The vectors v_i are the columns of a $k \times n$ -matrix A that represents the element of the Grassmannian. Then an element $V \in Gr_{kn}$ represented by A gives the matroid \mathcal{M}_V whose bases are the k -subsets $I \subset [n]$ such that $\Delta_I(A) \neq 0$. Here, $\Delta_I(A)$ denotes the determinant of A_I , the k by k submatrix of A in the column set I .

Then Gr_{kn} has a subdivision into *matroid strata* $S_{\mathcal{M}}$ labeled by some matroids \mathcal{M} :

$$S_{\mathcal{M}} := \{V \in Gr_{kn} \mid \mathcal{M}_V = \mathcal{M}\}$$

The elements of the stratum $S_{\mathcal{M}}$ are represented by matrices A such that $\Delta_I(A) \neq 0$ if and only if $I \in \mathcal{M}$.

Definition 1. The total order $<_w$ for $w \in S_n$ is defined as

$$a <_w b \text{ if } w^{-1}a < w^{-1}b \text{ for } a, b \in [n].$$

For $I, J \in \binom{[n]}{k}$, where

$$I = \{i_1, \dots, i_k\}, i_1 <_w i_2 < \cdots <_w i_k$$

and

$$J = \{j_1, \dots, j_k\}, j_1 <_w j_2 < \cdots <_w j_k.$$

Then we set

$$I \leq_w J \text{ if and only if } i_1 \leq_w j_1, \dots, i_k \leq_w j_k.$$

This ordering is called the Gale ordering on $\binom{[n]}{k}$ induced by w . We denote \leq_t for $t \in [n]$ as $<_{c^{t-1}}$ where $c = (1, \dots, n) \in S_n$ is the long cycle, then we get the same ordering as the one we defined in the introduction.

Remark 2. Fix $I, J \in \binom{[n]}{k}$ such that $I \leq J$. Draw a planar horizontal line graph with vertex set $\{I \setminus J\} \cup \{J \setminus I\}$, such that the labels increase as we go from left to right. Color elements of $I \setminus J$ with red and elements of $J \setminus I$ with blue. Then there is a unique way to connect each red vertex to a blue vertex such that the lines are pairwise non-crossing. Denote $I \setminus J = \{i_1, \dots, i_t\}$, $J \setminus I = \{j_1, \dots, j_t\}$ such that i_r is connected to j_r for $r \in [t]$. Then we write $J = (I, \{i_1 \rightarrow j_1, \dots, i_t \rightarrow j_t\})$ and denote each $i_r \rightarrow j_r$ as a *swap*. From the definition of the Gale ordering, we get $i_r \leq j_r$ for each $r \in [t]$.

We can define matroids from the Gale ordering. See [G],[BGW].

Definition 3. Let $\mathcal{M} \subseteq \binom{[n]}{k}$. Then \mathcal{M} is a matroid if and only if \mathcal{M} satisfies the following property. For every $w \in S_n$, the collection \mathcal{M} contains a unique member $A \in \mathcal{M}$ maximal in \mathcal{M} with respect to the partial order \leq_w .

Now we can define a *permuted Schubert matroid* using the partial order \leq_w .

Definition 4. For $I = (i_1, \dots, i_k)$, the *permuted Schubert Matroid* SM_I^w consists of bases $H = (j_1, \dots, j_k)$ such that $I \leq_w H$.

Let us define the totally nonnegative Grassmannian and its cells.

Definition 5 ([P], Definition 3.1). The *totally nonnegative Grassmannian* $Gr_{kn}^{tnn} \subset Gr_{kn}$ is the quotient $Gr_{kn}^{tnn} = GL_k^+ \backslash Mat_{kn}^{tnn}$, where Mat_{kn}^{tnn} is the set of real $k \times n$ -matrices A of rank k with nonnegative maximal minors $\Delta_I(A) \geq 0$ and GL_k^+ is the group of $k \times k$ -matrices with positive determinant.

Definition 6 ([P], Definition 3.2). *Totally nonnegative Grassmann cells* $S_{\mathcal{M}}^{tnn}$ in Gr_{kn}^{tnn} are defined as $S_{\mathcal{M}}^{tnn} := S_{\mathcal{M}} \cap Gr_{kn}^{tnn}$. \mathcal{M} is called a *positroid* if the cell $S_{\mathcal{M}}^{tnn}$ is nonempty.

Note that from above definitions, we get

$$S_{\mathcal{M}}^{tnn} = \{GL_k^+ \bullet A \in Gr_{kn}^{tnn} \mid \Delta_I(A) > 0 \text{ for } I \in \mathcal{M}, \Delta_I(A) = 0 \text{ for } I \notin \mathcal{M}\}.$$

In [P], Postnikov showed a bijection between each cell and a combinatorial object called Grassmann necklace.

Definition 7 ([P], Definition 16.1). A *Grassmann necklace* is a sequence $\mathcal{I} = (I_1, \dots, I_n)$ of subsets $I_r \subseteq [n]$ such that, for $i \in [n]$, if $i \in I_i$ then $I_{i+1} = (I_i \setminus \{i\}) \cup \{j\}$, for some $j \in [n]$; and if $i \in I_i$ then $I_{i+1} = I_i$. (Here the indices are taken modulo n .) In particular, we have $|I_1| = \dots = |I_n|$.

An example of a Grassmann necklace would be $I_1 = \{1, 2, 4\}, I_2 = \{2, 4, 5\}, I_3 = \{3, 4, 5\}, I_4 = \{4, 5, 2\}, I_5 = \{5, 1, 2\}$.

Two of the results in [P] are the following:

Lemma 8 ([P], Lemma 16.3). *For a matroid $\mathcal{M} \subseteq \binom{[n]}{k}$ of rank k on the set $[n]$, let $\mathcal{I}_{\mathcal{M}} = (I_1, \dots, I_n)$ be the sequence of subsets such that I_i is the minimal member of \mathcal{M} with respect to \leq_i . Then $\mathcal{I}_{\mathcal{M}}$ is a Grassmann necklace.*

Theorem 9 ([P], Theorem 17.2). *Let $S_{\mathcal{M}}^{tnn}$ be a nonnegative Grassmann cell, and let $\mathcal{I}_{\mathcal{M}} = (I_1, \dots, I_n)$ be the Grassmann necklace corresponding to \mathcal{M} . Then*

$$S_{\mathcal{M}}^{tnn} = \bigcap_{i=1}^n \Omega_{I_i}^{c^{i-1}} \cap Gr_{kn}^{tnn}$$

where $c = (1, \dots, n) \in S_n$ and $\Omega_{I_i}^{c^{i-1}}$ is the permuted Schubert cell, which is the set of elements $V \in Gr_{kn}$ such that I_i is the lexicographically minimal base of M_V with respect to ordering $<_w$ on $[n]$.

These results imply that bases of a positroid are included in each cyclically shifted Schubert matroids corresponding to a corresponding Grassmann necklace. But it does not imply that they are equal. Postnikov therefore conjectured that each positroid is exactly the intersection of cyclically shifted Schubert matroids. This is what we are going to prove in our paper:

Theorem 10. \mathcal{M} is a positroid if and only if for some Grassmann necklace (I_1, \dots, I_n) ,

$$\mathcal{M} = \bigcap_{i=1}^n SM_{I_i}^{c^{i-1}}.$$

In other words, \mathcal{M} is a positroid if and only if the following holds : $H \in \mathcal{M}$ if and only if $H \geq_t I_t$ for any $t \in [n]$.

3. J-DIAGRAMS AND J-GRAPHS

In [P], Postnikov also showed a bijection between positroids and combinatorial objects called J-diagrams. Let's define fillings of a Young diagram.

Definition 11. Fix a partition λ that fit inside the rectangle $(n - k)^k$. The boundary of the Young diagram of λ gives the lattice path of length n from the upper right corner to the lower left corner of the rectangle $(n - k)^k$. Label each edges in the path by $1, \dots, n$ as we go downwards and to the left. Define $I(\lambda)$ as the set of lables of k vertical steps in the path.

Each column and row contains exactly one labeled edge. We will index the columns and rows with it. Then we will say that a box is at (i, j) if it is on row i and column j . A *filling* of λ is a diagram of λ where each box is either empty or filled with a dot. Given a filling L , we define sets $b(L), f(L), e(L), c(L)$ as

$$\begin{aligned} b(L) &:= \{(i, j) \mid \text{there is a box at } (i, j) \text{ in } L\}, \\ f(L) &:= \{(i, j) \mid \text{there is a box filled with a dot at } (i, j) \text{ in } L\}, \\ e(L) &:= b(L) \setminus f(L), \\ c(L) &:= \{(i, j) \mid i \in I(\lambda), j \in ([n] \setminus I(\lambda)) \cup \{0\}\}. \end{aligned}$$

The set $c(L)$ is obtained by attaching a column labeled 0 to the right of the rectangle $(n - k)^k$. Now we are ready to define J-diagrams.

Definition 12 ([P], Definition 6.1). For a partition λ , let us define a J-diagram L of shape λ as a filling of boxes of the Young diagram of shape λ such that, for any three boxes indexed $(i, j), (i', j), (i, j')$, where $i' < i$ and $j' < j$, if boxes on position (i', j) and (i, j') are filled, then the box on (i, j) is also filled. This property is called the J-property. We will say that a J-diagram is *full* if every box is filled.

Fix a J-diagram L of shape λ . For a $(i, j) \in c(L)$, we denote the NW-region of it to be

$$NW_{(i,j)} := \{(i', j') \mid i' < i, j' > j\}.$$

If there is a dot in this region, then there is a unique dot (i', j') that minimizes $i - i'$ and $j' - j$ at the same time, due to the J-property. Let's denote such (i', j') by $cov(i, j)$.

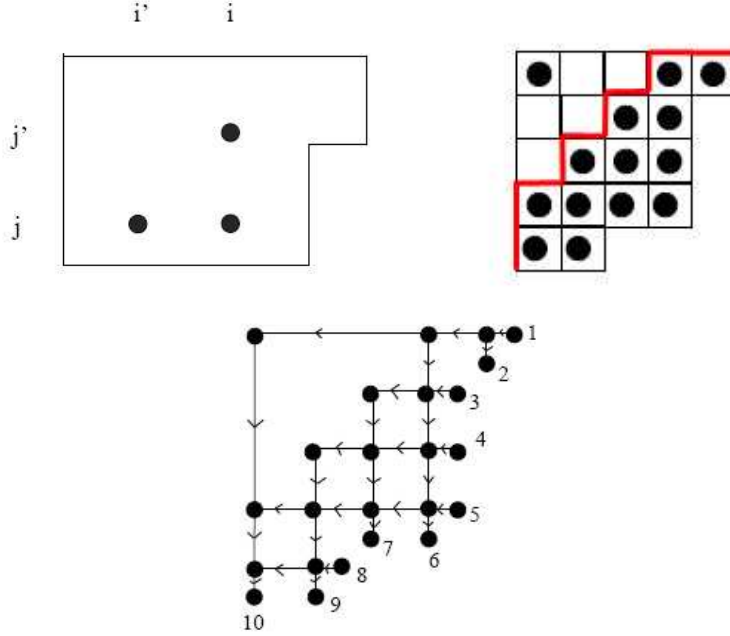


FIGURE 1. \lrcorner -property and an example of a \lrcorner -diagram and a \lrcorner -graph

Definition 13. Fix $(i, j) \in f(L)$. Then let us say that the dot at (i, j) is covered by the dot at $(i', j') = \text{cov}(i, j)$ and denote $(i', j') \triangleleft (i, j)$.

Given a \lrcorner -diagram, put dots on each edge of the boundary path. And connect all dots on same row and connect all dots on the same column. After orienting the edges as in Figure 1, we get a \lrcorner -graph.

Definition 14 ([P]). \lrcorner -graph is obtained from a \lrcorner -diagram in the following way. Place a vertex in the middle of each step in the boundary lattice path of the diagram and mark these vertices by $1, 2, \dots, n$. We will call these vertices the *boundary vertices*. Now for each dot inside the \lrcorner -diagram, draw a horizontal line to its right, and vertical line to its bottom until it reaches the boundary of the diagram. Then orient all vertical edges downward and horizontal edges to the left.

The source set of the \lrcorner -graph is given by $I(\lambda)$ and the sink set is given by $[n] \setminus I(\lambda)$.

Definition 15. A *path* in a \lrcorner -graph is a directed path that starts at some boundary vertex and ends at some boundary vertex. Given a path p , we denote its starting point and end point by p^s and p^e . A *family of non-touching paths* is a family of paths where no two paths share a vertex in the network.

Given a family of non-touching paths $\{p_1, \dots, p_t\}$, we say that this family represents $J = I(\lambda) \setminus \{p_1^s, \dots, p_t^s\} \cup \{p_1^e, \dots, p_t^e\}$. Empty family is also considered as a valid family. Now the following proposition follows as a corollary of ([P], Theorem 6.5).

Proposition 16 ([P]). Fix a positroid \mathcal{M}_L that corresponds to a \lrcorner -diagram L . Then $J \in \mathcal{M}_L$ if and only if J is represented by a family of non-touching paths in the \lrcorner -graph of a \lrcorner -diagram L .

It is obvious that $I_1 = I(\lambda)$. Now let's try to read off $\mathcal{I} = (I_1, \dots, I_n)$ directly from L .

For each $(i, j) \in f(L)$, we get a unique chain $(i_t, j_t) \triangleleft \dots \triangleleft (i_1, j_1) = (i, j)$ such that (i_t, j_t) is not covered by any other dot in the diagram. Call this a chain at (i, j) . For each (i_r, j_r) , look at a path that starts at i_r , goes left to (i_r, j_r) and then comes down to end at j_r . The collection of such paths forms a family of non-touching paths. So we get $J_{(i,j)} := (I(\lambda), \{i_1 \rightarrow j_1, \dots, i_t \rightarrow j_t\}) \in \mathcal{M}_L$. We can also define this for all $(i, j) \in c(L) \setminus f(L)$, by looking at the chain $(i_t, j_t) \triangleleft \dots \triangleleft (i_1, j_1)$ such that $(i_1, j_1) = \text{cov}(i, j)$.

Proposition 17. *Fix a \mathcal{J} -diagram of shape λ and let $\mathcal{I} = (I_1, \dots, I_n)$ be the Grassmann necklace of \mathcal{M}_L . For $j \notin I(\lambda)$, let (a, j) be the lowest dot in column j . Then $I_j = J_{(a,j)}$. If no such dot exists, then $I_j = I_{j+1}$. For $j \in I(\lambda)$, let (j, b) be the dot that contains the boundary edge labeled j . Now let (j, b') be the box right of (j, b) in $c(L)$. Then $I_j = J_{(j,b')}$.*

Proof. Let \mathcal{F}_j be the family of non-touching paths that represent I_j . Then \mathcal{F}_j cannot contain any path p such that $p^s, p^e < j$ or $p^s, p^e > j$. Because if it does contain such path, then $\mathcal{F}_j \setminus \{p\}$ represents J such that $J <_j I_j$.

Let's start with the case when $j \notin I(\lambda)$. If no such (a, j) as given in the proposition exists, it means j is either a loop or a coloop in \mathcal{M}_L . So it follows that $I_j = I_{j+1}$. Now assume that such dot exists in L . Let the chain at (a, j) be $(i_t, j_t) \triangleleft \dots \triangleleft (i_1, j_1) = (a, j)$. Then \mathcal{F}_j must contain the path $p_1 := i_1 \rightarrow j_1$. If not, then $\mathcal{F}_j \not\leq_j I(\lambda) \setminus \{i_1\} \cup \{j_1\}$. This is because for all $p \in \mathcal{F}_j$, from the fact that $p^s < j < p^e$, it follows that $p^s \leq i_1$ and $p^e \geq j_1$. Now that we know $p_1 \in \mathcal{F}_j$, we can also find out that \mathcal{F}_j must contain the path $p_2 := i_2 \rightarrow j_2$. Because if not, we get $\mathcal{F}_j \not\leq_j I(\lambda) \setminus \{i_1, i_2\} \cup \{j_1, j_2\}$ due to the fact that for any path $p \in \mathcal{F}_j \setminus \{p_1\}$, $p^s \leq i_2$ and $p^e \geq j_2$. So repeating this process, we get that $\mathcal{F}_j = \{i_1 \rightarrow j_1, \dots, i_t \rightarrow j_t\}$. Hence we obtain $I_j = J_{(a,j)}$.

Now look at the case when $j \in I(\lambda)$. If there are not dots in $nw_{(j,b')}$, then there can be no paths p such that $p^s < j < p^e$. So we get $I_j = I(\lambda)$. Now assume that the chain at $nw_{(j,b')}$ is given by $(i_t, j_t) \triangleleft \dots \triangleleft (i_1, j_1) = \text{cov}(j, b')$. Then \mathcal{F}_j must contain the path $p_1 := i_1 \rightarrow j_1$. If not, then $\mathcal{F}_j \not\leq_j I(\lambda) \setminus \{i_1\} \cup \{j_1\}$. This is because for all $p \in \mathcal{F}_j$, from the fact that $p^s < j < p^e$, it follows that $p^s \leq i_1$ and $p^e \geq j_1$. So we follow the exactly same process as the previous case when $j \notin I(\lambda)$, to obtain $\mathcal{F}_j = \{i_1 \rightarrow j_1, \dots, i_t \rightarrow j_t\}$ and hence $I_j = J_{(j,b')}$. \square

Let's look at an example. In the \mathcal{J} -diagram in Figure 1, I_4 is given by $J_{(4,2)}$. Chain at $(4, 2)$ is given by $(1, 10) \triangleleft (3, 7) \triangleleft (4, 6)$. So $I_4 = I_1 \setminus \{1, 3, 4\} \cup \{10, 7, 6\} = \{5, 6, 7, 8, 10\}$. I_9 is given by $J_{(8,9)}$. Chain at $(8, 9)$ is given by $(5, 10) \triangleleft (8, 9)$. So $I_9 = I_1 \setminus \{5, 8\} \cup \{9, 10\} = \{1, 3, 4, 8, 10\}$.

4. PROOF OF THE MAIN THEOREM

In this section, we will prove the main theorem by showing that for each Grassmann necklace $\mathcal{I} = (I_1, \dots, I_n)$ and a positroid $\mathcal{M}_{\mathcal{I}}$, we have $\bigcap_{i=1}^n SM_{I_i}^{c_i-1} \subseteq \mathcal{M}_{\mathcal{I}}$. To do this, we need to show that each $J \in \bigcap_{i=1}^n SM_{I_i}^{c_i-1}$ can be expressed as a family of non-touching paths inside the \mathcal{J} -graph of $\mathcal{M}_{\mathcal{I}}$. In order to accomplish this, we will start from a full- \mathcal{J} -diagram and do an induction type of argument by increasing the number of empty boxes.

Let's first show that for a full \mathcal{J} -diagram L , the positroid \mathcal{M}_L is a Schubert matroid.

Lemma 18. *Let L be a full \mathcal{J} -diagram of shape λ . Then $\mathcal{M}_L = SM_{I(\lambda)}$.*

Proof. We need to show that for all $J \in SM_{I(\lambda)}$, $J \in \mathcal{M}_L$. Fix a $J \in SM_{I(\lambda)}$. We can express J as series of non-crossing swaps $(I(\lambda), \{i_1 \rightarrow j_1, \dots, i_r \rightarrow j_r\})$. For each swap $i_t \rightarrow j_t$, we associate a path inside the \mathcal{J} -graph by a path starting at i_t , going to (i_t, j_t) , then ending at j_t . Then we get a family of non-touching paths representing J . \square

Now let's show that given any \mathcal{J} -diagram $L_{\mathcal{I}}$ with Grassmann necklace $\mathcal{I} = (I_1, \dots, I_n)$, we can add a dot to obtain a \mathcal{J} -diagram $L_{\mathcal{I}'}$ such that writing $\mathcal{I}' = (I'_1, \dots, I'_n)$, there exists $\alpha \in [n]$ such that $I'_i = I_i$ for all $i \neq \alpha$ and $|I_\alpha \setminus I'_\alpha| = 1$.

First consider the case when there exists an empty box on the boundary strip of $L_{\mathcal{I}}$. Adding a dot to a box adjacent to the boundary line changes exactly one element of the grassmann necklace. If all such boxes are filled with dots, then adding a dot to any remaining box on the boundary strip will work.

So we will consider the case when all the boxes of boundary strip are filled. Now let's define the middle path of $L_{\mathcal{I}}$ to be a lattice path inside the diagram such that

- (1) all boxes between the middle path and the border path are filled with dots and
- (2) the inside corners of the upper region is empty. Here, *upper region* is the diagram obtained by looking at the boxes above the middle path and *inside corner* is a box such that boxes below and to the right of that box is not in the diagram.

Then putting a dot into any inside corner of the upper region will work. Example of a middle path is given as a red line in Figure 1.

Proposition 19. *Given any Grassmann necklace $\mathcal{I} = (I_1, \dots, I_n)$, $\mathcal{M}_{\mathcal{I}} = \bigcap_{i=1}^n SM_{I_i}^{c_i-1}$.*

Proof. Denote the number of empty boxes in a \mathcal{J} -diagram by m . We will prove the proposition by induction on m . When $m = 0$, this is the full \mathcal{J} -diagram case. Now let $\mathcal{M}_{\mathcal{I}}$ be any positroid such that $L_{\mathcal{I}}$ has m empty boxes. Use the construction above to obtain $L_{\mathcal{I}'}$, where $\mathcal{I}' = (I'_1, \dots, I'_n)$ and there exists $\alpha \in [n]$ such that $I'_i = I_i$ for all $i \neq \alpha$ and $|I_\alpha \setminus I'_\alpha| = 1$. For the sake of induction, let's assume that we know $\mathcal{M}_{\mathcal{I}'} = \bigcap_{i=1}^n SM_{I'_i}^{c_i-1}$. Now it is enough to show $\mathcal{M}_{\mathcal{I}'} \setminus \mathcal{M}_{\mathcal{I}} \subset SM_{I_\alpha}^{c_\alpha-1} \setminus SM_{I'_\alpha}^{c_\alpha-1}$ to show $\bigcap_{i=1}^n SM_{I_i}^{c_i-1} \subset \mathcal{M}_{\mathcal{I}}$.

Let the chain of dots $(w_{q+r}, z_{q+r}) \triangleleft \dots \triangleleft (w_q, z_q) \triangleleft \dots \triangleleft (w_1, z_1)$ represent I'_α such that (w_q, z_q) is the newly added dot going from $L_{\mathcal{I}}$ to $L_{\mathcal{I}'}$. Then any family of non-touching paths representing some $J \in \mathcal{M}_{\mathcal{I}'} \setminus \mathcal{M}_{\mathcal{I}}$ should use (w_q, z_q) as a NW-corner. Here, *NW-corner* of a path means a dot at (i, j) such that in the \mathcal{J} -graph, the path comes into (i, j) by a horizontal edge and heads out by a vertical edge. Let's fix such J and a family.

Denote the path going through (w_q, z_q) by p_q . Since p_q uses (w_q, z_q) as a NW-corner, if there is a path going through (w_{q-1}, z_{q-1}) , it must use (w_{q-1}, z_{q-1}) as a NW-corner too. Assume there is no such path. Then since there are points $(w_q, z_{q-1}), (w_{q-1}, z_q)$ inside the diagram, we can perturb the path p_q to go through points $(w_q, z_{q-1}), (w_{q-1}, z_{q-1}), (w_{q-1}, z_q)$. Then we get a family of non-touching paths that still represent J but do not go through (w_q, z_q) and get a contradiction.

So we see that there is a path inside the family that uses (w_{q-1}, z_{q-1}) as a NW-corner. Repeating this argument, we see that the family contains p_q, p_{q-1}, \dots, p_1 each having $(w_q, z_q), \dots, (w_1, z_1)$ as a NW-corner.

Now what is the chain of dots that represent I_α ? There can be three cases. Let's show $J \not\leq_\alpha I_\alpha$ in each of those cases.

- (1) When I_α is represented by $(w_{q+r}, z_{q+r}) \triangleleft \dots \triangleleft (w_{q+1}, z_{q+1}) \triangleleft (w_{q-1}, z_{q-1}) \dots \triangleleft (w_1, z_1)$:

Take a look at p_q . If $p_q^e <_\alpha z_{q+1}$ or $p_q^s >_\alpha w_{q+1}$ then we have $J \not\geq_\alpha I_\alpha$ and is finished. So let's assume $p_q^e \geq_\alpha z_{q+1}$ and $p_q^s \leq_\alpha w_{q+1}$. If there is no path going through (w_{q+1}, z_{q+1}) in the family, then the path p_q can be slightly changed so that we get a family of non-touching paths representing J and not using (w_q, z_q) . So there is a path p_{q+1} in the family going through (w_{q+1}, z_{q+1}) . Now take a look at p_{q+1} . Repeating the above argument for p_{q+1} , we get there is a path going through (w_{q+2}, z_{q+2}) in the family. Repeating this process, we see that there are paths going through each of the (w_i, z_i) 's. But this tells us that $\{p_{q+r}^e, \dots, p_1^e\} \subset J$ and hence $J \not\geq_\alpha I_\alpha$.

- (2) When I_α is represented by $(w_{q+s}', z_{q+s}') \cdots \triangleleft (w_{q+1}', z_{q+1}') \triangleleft (w_q', z_q) \triangleleft (w_{q-1}, z_{q-1}) \cdots \triangleleft (w_1, z_1)$:

Take a look at p_q . If $p_q^s >_\alpha w_q'$ then we have $J \not\geq_\alpha I_\alpha$ and is finished. So let's assume $p_q^s \leq_\alpha w_q'$. If there is no path in the family going through (w_q', z_q) , then the path p_q can be slightly changed to get a path going through (w_q', z_q) and still not touch other paths in the family. Then this newly obtained family represents J but none of its paths use (w_q, z_q) as a NW-corner. Hence we get a contradiction. So see that there is a path p_{q+1} going through (w_q', z_q) in the family.

Now take a look at p_{q+1} . If $p_{q+1}^e <_\alpha z_{q+1}'$ or $p_{q+1}^s >_\alpha w_{q+1}'$ then we have $J \not\geq_\alpha I_\alpha$ and is finished. Imitating the proof of case (1), we see that there is a path p_{q+2} in the family going through (w_{q+1}', z_{q+1}') . Repeating this argument as in (1), we obtain $\{p_{q+s+1}^e, \dots, p_1^e\} \subset J$ and hence $J \not\geq_\alpha I_\alpha$.

- (3) When I_α is represented by $\cdots \triangleleft (w_{q+1}, z_{q+1}) \triangleleft (w_q, z_q') \triangleleft (w_{q-1}, z_{q-1}) \cdots \triangleleft (w_1, z_1)$:

The proof is similar to case (2) and is omitted.

So we have shown $\mathcal{M}_{\mathcal{I}'} \setminus \mathcal{M}_{\mathcal{I}} \subset SM'_{I_\alpha}{}^{c_\alpha-1} \setminus SM_{I_\alpha}{}^{c_\alpha-1}$, and we are finished. □

5. EXAMPLES

Now we will show an example of the usefulness of the main theorem for explicitly computing bases of a positroid. Let \mathcal{M} be a positroid indexed by a decorated permutation $[5, 3, 2, 1, 4]$. The function col wouldn't matter since we don't have a fixed point.

$$I_1 = \{1, 2, 4\}$$

$$I_2 = \{2, 4, 5\}$$

$$I_3 = \{3, 4, 5\}$$

$$I_4 = \{4, 5, 2\}$$

$$I_5 = \{5, 1, 2\}$$

$$\begin{aligned} \mathcal{M} &= \{H \mid H \geq_1 I_1, H \geq_2 I_2, \dots, H \geq_5 I_5\} \\ &= \{\{1, 2, 4\}, \{1, 2, 5\}, \{1, 3, 4\}, \{1, 3, 5\}, \{2, 4, 5\}, \{3, 4, 5\}\} \end{aligned}$$

Now let \mathcal{M} be a positroid indexed by a decorated permutation $[5, 3, 2, 1, 4, 6]$, with $col(6) = 1$. Then \mathcal{M} is same as above. If $col(6) = -1$, then we get:

$$I_1 = \{1, 2, 4, 6\}$$

$$I_2 = \{2, 4, 5, 6\}$$

$$I_3 = \{3, 4, 5, 6\}$$

$$I_4 = \{4, 5, 6, 2\}$$

$$I_5 = \{5, 6, 1, 2\}$$

$$I_6 = \{1, 2, 4, 6\}$$

$$\mathcal{M} = \{\{1, 2, 4, 6\}, \{1, 2, 5, 6\}, \{1, 3, 4, 6\}, \{1, 3, 5, 6\}, \{2, 4, 5, 6\}, \{3, 4, 5, 6\}\}.$$

6. DECORATED PERMUTATIONS AND THE UPPER GRASSMANN NECKLACE

In this section we will show that a positroid is an intersection of permuted dual Schubert matroids. The tools developed here will also be used for expressing lattice path matroids with a Grassmann necklace in the next section.

Definition 20 ([P], Definition 13.3). A decorated permutation $\pi^\cdot = (\pi, \text{col})$ is a permutation $\pi \in S_n$ together with a coloring function col from the set of fixed points $\{i | \pi(i) = i\}$ to $\{1, -1\}$. That is, a decorated permutation is a permutation with fixed points colored in two colors.

It is easy to see the bijection between necklaces and decorated permutations. To go from a Grassmann necklace \mathcal{I} to a decorated permutation $\pi^\cdot = (\pi, \text{col})$.

- if $I_{i+1} = (I_i \setminus \{i\}) \cup \{j\}$, $j \neq i$, then $\pi(i) = j$
- if $I_{i+1} = I_i$ and $i \notin I_i$ then $\pi(i) = i$, $\text{col}(i) = 1$
- if $I_{i+1} = I_i$ and $i \in I_i$ then $\pi(i) = i$, $\text{col}(i) = -1$.

To go from a decorated permutation $\pi^\cdot = (\pi, \text{col})$ to a Grassmann necklace \mathcal{I} ,

$$I_r = \{i \in [n] | i <_r \pi^{-1}(i) \text{ or } (\pi(i) = i \text{ and } \text{col}(i) = -1)\}.$$

Let's look at a simple example. For decorated permutation π^\cdot with $\pi = 81425736$ and $\text{col}(5) = 1$, we get $I_1 = \{1, 2, 3, 6\}$, $I_2 = \{2, 3, 6, 8\}$, $I_3 = \{3, 6, 8, 1\}$, $I_4 = \{4, 6, 8, 1\}$, $I_5 = \{6, 8, 1, 2\}$, $I_6 = \{6, 8, 1, 2\}$, $I_7 = \{7, 8, 1, 2\}$, $I_8 = \{8, 1, 2, 3\}$.

Let's look at the definition of a dual Schubert matroid.

Definition 21. Fix a base set $[n]$ and $w \in S_n$. For $I = (i_1, \dots, i_k) \in \binom{[n]}{k}$, the *dual Schubert matroid* \tilde{M}_I^w consists of bases $H = (j_1, \dots, j_k)$ such that $I \leq_w H$.

Fix a decorated permutation $\pi^\cdot = (\pi, \text{col})$. Let $\mathcal{I}_{\pi^\cdot} = (I_1, \dots, I_n)$ be the corresponding Grassmann necklace and \mathcal{M}_{π^\cdot} the corresponding positroid.

Lemma 22. For all $H \in \mathcal{M}_{\pi^\cdot}$, $H \leq_i \pi^{-1}(I_i)$ for all $i \in [n]$.

Proof. We can assume π has no fixed point since they correspond to loops or coloops of \mathcal{M}_{π^\cdot} . Denote $I_1 = \{j_1, \dots, j_k\}$ where j_1, \dots, j_k are labeled so that the following condition is satisfied:

$$\pi^{-1}(j_1) < \pi^{-1}(j_2) < \dots < \pi^{-1}(j_k).$$

Notice that for any $r \in [k]$ and $t \in (\pi^{-1}(j_r), 1]$, we have $j_r \in I_t$. So for $t \in (\pi^{-1}(j_i), \pi^{-1}(j_{i+1}))$, $\{j_1, \dots, j_i\} \subset I_t$. Denote elements of H by $x_1 < x_2 < \dots < x_k$. Let i be the biggest in $[k]$ such that

- (1) $x_t \leq \pi^{-1}(j_t)$ for all $t \in [i+1, k]$ and
- (2) $x_i > \pi^{-1}(j_i)$.

We have $|H \cap [1, x_i - 1]| < i$. But since $j_k < \pi^{-1}(j_k) < t$ for all $t \in [1, i]$, $|I_t \cap [1, x_i - 1]| \geq i$. Then this contradicts $H \geq_{x_i} I_{x_i}$. So there cannot be such i that satisfies the above condition, so $H \leq \{\pi^{-1}(j_1), \dots, \pi^{-1}(j_k)\}$. Similar for other \leq_i 's. \square

Now look at $(J_1 := \pi^{-1}(I_1), \dots, J_n := \pi^{-1}(I_n))$. They form a necklace in the sense that $J_{i+1} = J_i \setminus \{\pi^{-1}(i)\} \cup \{i\}$ except for i such that $\pi(i) = i$. We will call this the *upper Grassmann necklace* of π .

To go from a decorated permutation $\pi^\cdot = (\pi, \text{col})$ to a Grassmann necklace \mathcal{J} ,

$$J_r = \{i \in [n] \mid \pi(i) <_r i \text{ or } (\pi(i) = i \text{ and } \text{col}(i) = -1)\}.$$

Define $\tilde{\mathcal{M}}$ as:

$$\tilde{\mathcal{M}} = \bigcap_{i=1}^n SM_{J_i}^{c^{i-1}}.$$

Then Lemma 22 tells us that $\mathcal{M} \subseteq \tilde{\mathcal{M}}$. The proof of the following lemma is similar to Lemma 22.

Lemma 23. *For any $H \in \tilde{\mathcal{M}}$, $H \geq_i \pi(J_i) = I_i$ for all $i \in [n]$.*

So we have the following theorem.

Theorem 24. *Pick a decorated permutation. π^\cdot . Then we have the corresponding Grassmann necklace and an upper Grassmann necklace, $\mathcal{I} = (I_1, \dots, I_n)$, $\mathcal{J} = (J_1, \dots, J_n)$. Then $J_i = \pi^{-1}(I_i)$ for all $i \in [n]$ where $\pi^\cdot = (\pi, \text{col})$. And we have the equality*

$$\bigcap_{i=1}^n SM_{I_i}^{c^{i-1}} = \bigcap_{i=1}^n SM_{J_i}^{c^{i-1}}.$$

7. LATTICE PATH MATROIDS

Lattice path matroids were defined in [BMN]. These are very simple cases of positroids. In this section we will show a simple way to get a decorated permutation corresponding to a given lattice Path matroid.

Definition 25. *Lattice path matroids* are defined as the following. Pick a base set $[n]$ and $I, J \in \binom{[n]}{k}$ such that $I \leq J$.

$$LP_{I,J} = \{H \mid H \in \binom{[n]}{k}, I \leq H \leq J\} = SM_I \cap \tilde{SM}_J$$

Since I, J corresponds to two lattice paths in a n -by- k grid, $LP_{I,J}$ expresses all the lattice paths between them. Let's prove that a lattice path matroid is a positroid.

Lemma 26. *Any lattice path matroid is a positroid.*

Proof. Denote the base set by $[n]$. Let $I = \{a_1, \dots, a_k\}$, $J = \{b_1, \dots, b_k\}$ such that $a_1 < \dots < a_k$, $b_1 < \dots < b_k$, $I \leq J$. Let's prove $LP_{I,J}$ is a positroid by constructing a k -by- n matrix such that $\Delta_H = 0$ for all $H \in \binom{[n]}{k} \setminus LP_{I,J}$ and $\Delta_H > 0$ for all $H \in LP_{I,J}$.

Let $V = (v_{ij})_{i,j=1,1}^{k,n}$ be a k -by- n Vandermonde matrix. Set $v_{ij} = 0$ for all $j \notin [a_i, b_i]$. So V would look like

$$v_{ij} = \begin{cases} x_i^{j-1} & \text{if } a_i \leq j \leq b_i \\ 0 & \text{otherwise} \end{cases}.$$

Now set values of x_1, \dots, x_k such that $x_1 > 1$ and $x_{i+1} = x_i^{k^2}$ for all $i \in [k-1]$. Let's denote $V_{[1..i], [c_1, \dots, c_i]}$ as a submatrix of V by taking rows from 1 to i and columns c_1, \dots, c_i . Then by construction of V , for all $1 < i \leq m$, $\Delta_{V_{[1..i], [c_1, \dots, c_{i-1}, c_i]}} > 0$ if and only if v_{i, c_i} nonzero and $\Delta_{V_{[1..i-1], [c_1, \dots, c_{i-1}]}} > 0$. So $\Delta_H > 0$ if and only if $V_{[1..k], H}$ has nonzero diagonal entries. And that happens if and only if $H \in LP_{I, J}$. So we have proven that $LP_{I, J}$ is a positroid. \square

Now fix a base set $[n]$. Choose any $I = \{a_1, \dots, a_k\}, J = \{b_1, \dots, b_k\}$ in $\binom{[n]}{k}$ such that $a_1 < \dots < a_k, b_1 < \dots < b_k, I \leq J$. So we have chosen a lattice path matroid $LP_{I, J}$. Let's try to find π^\cdot that corresponds to $LP_{I, J}$. Denote $I = (i_1, \dots, i_k), J = (j_1, \dots, j_k)$ written in increasing order. If $i_t = j_t$ for some $t \in [k]$, this corresponds to a fixed point of π^\cdot with $col(i_t) = -1$. Let's assume that this doesn't occur for convenience, hence assuming that π has no fixed points. Then we have have

- $I = \{i \in [n] | i < \pi^{-1}(i)\}, J = \{i \in [n] | \pi(i) < i\}$ and
- $\pi(J) = I, \pi([n] - J) = [n] - I$.

So if π satisfies the following properties:

- (1) $\pi(J) = I, \pi([n] - J) = [n] - I$
- (2) For all $j \in J, \pi(j) < j$
- (3) For all $j \in [n] - J, \pi(j) > j$,

then the corresponding positroid \mathcal{M}_π is contained in $LP_{I, J}$.

Denote by $P_{I, J}$ the subset of S_n consisting of permutations satisfying above properties. Let τ_{ij} stand for the transposition of i and j .

Lemma 27. *Pick any $a, b \in J$ such that $a < b$. Assume we have $\pi \in P_{I, J}$ with $\tau_{ab}\pi \in P_{I, J}$. If $\pi(a) < \pi(b)$ then $\mathcal{M}_{\tau_{ab}\pi} \subset \mathcal{M}_\pi$. Similarly, pick $c, d \in [n] - J, c < d$. Assume we have $\pi \in P_{I, J}$ with $\tau_{cd}\pi \in P_{I, J}$. If $\pi(c) < \pi(d)$ then $\mathcal{M}_{\tau_{cd}\pi} \subset \mathcal{M}_\pi$.*

Proof. If $\pi(a) < \pi(b)$, then $\pi \in P_{I, J}$ and $\tau_{ab}\pi \in P_{I, J}$ gives us $a < b < \pi(a) < \pi(b)$. So if we have a Grassmann necklace $I_\pi = (I_1, \dots, I_n)$ corresponding to π , $I_{\tau_{ab}\pi}$ is obtained from I_π by changing all $\pi(a)$'s in I_a, \dots, I_{b-1} to $\pi(b)$. Therefore $\mathcal{M}_{\tau_{ab}\pi} \subset \mathcal{M}_\pi$. Similarly we can show $\mathcal{M}_{\tau_{cd}\pi} \subset \mathcal{M}_\pi$. \square

It follows that $\pi \in P_{I, J}$ corresponds to the biggest positroid under inclusion that satisfies the following:

- For all $a, b \in J$ such that $a < b, \pi(a) < \pi(b)$
- For all $c, d \in [n] - J$ such that $c < d, \pi(c) < \pi(d)$.

Combining this fact with Lemma 26, we have the following theorem.

Theorem 28. *Pick a base set $[n]$, $I = \{i_1, \dots, i_k\}, J = \{j_1, \dots, j_k\} \in \binom{[n]}{k}$ such that $i_1 < \dots < i_k, j_1 < \dots < j_k$ and $I \leq J$. Then $LP_{I, J}$ is a positroid and corresponds to the decorated permutation $\pi^\cdot = (\pi, col)$ defined as the follows:*

$$\begin{aligned} \pi(j_r) &= i_r \text{ for all } r \in [k] \\ \pi(d_r) &= c_r \text{ for all } r \in [n - k] \end{aligned}$$

$$\text{If } \pi(t) = t \text{ then } \text{col}(t) = \begin{cases} -1 & \text{if } t \in J \\ 1 & \text{otherwise} \end{cases}$$

where $[n] \setminus J = \{d_1, \dots, d_{n-k}\}$, $[n] \setminus I = \{c_1, \dots, c_{n-k}\}$ such that $d_1 < \dots < d_{n-k}$, $c_1 < \dots < c_{n-k}$.

8. FURTHER REMARKS

An interesting problem would be to describe the circuits of a positroid in terms of circuits of permuted Schubert matroids. Let's recall the definition of the circuits of a matroid.

Definition 29. Given a matroid \mathcal{M} on $[n]$, a subset of $[n]$ is called independent if it is a subset of some $I \in \mathcal{M}$, and dependent otherwise. Then a minimum dependent set with respect to inclusion is called a circuit of \mathcal{M} . $C(\mathcal{M})$ will stand for the set of circuits of \mathcal{M} .

The following problem was proposed by Allen Knutson:

Problem 30. *Following notation of Theorem 10, can one describe the circuits of \mathcal{M} directly from circuits of $SM_{I_1}, SM_{I_2}^c, \dots, SM_{I_n}^{c^{n-1}}$?*

We could set $C(\mathcal{M})' := \bigcup_{i=1}^n C(SM_{I_i}^{c^{i-1}})$. From the definition of circuits above in terms of minimum dependent sets, we could choose minimal sets with respect to inclusion in C' to get C . But it appears that although each set contained in C' contains a circuit of \mathcal{M} as a subset, some are not the circuits of \mathcal{M} . It would also be interesting to find out for which decorated permutations $C(\mathcal{M})$ and $C'(\mathcal{M})$ are equal.

Now as positroids correspond to matroid strata of the nonnegative part of the Grassmannian, we could try to generalize it. Flag matroids correspond to the matroid strata of a flag variety. Let $[n]$ be the base set as before.

Definition 31. A *flag* F is a strictly increasing sequence

$$F^1 \subset F^2 \subset \dots \subset F^m$$

of finite sets. Denote by k_i the cardinality of the set F^i . We write $F = (F^1, \dots, F^m)$. The set F^i is called the i -th constituent of F .

Theorem 32 ([BGW]). *A collection \mathcal{F} of flags of rank (k_1, \dots, k_m) is a flag matroid if and only if*

- (1) *For all $i \in [m]$, M_i the collection of F^i 's for each $F \in \mathcal{F}$ form a matroid.*
- (2) *For every $w \in S_n$, the \leq_w -minimal bases of each M_i form a flag. If this holds, we say that M_i 's are concordant.*
- (3) *Every flag*

$$B_1 \subset \dots \subset B_m$$

such that B_i is a basis of M_i for $i = 1, \dots, m$ belongs to \mathcal{F} .

Definition 33. A *flag positroid* is a flag matroid in which all constituents are positroids.

It would be interesting to see what are the necessary conditions for two decorated permutations so that their corresponding positroids are concordant.

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